

LOCAL WELL-POSEDNESS FOR THE (3+1) - DIMENSIONAL MAXWELL-KLEIN-GORDON EQUATIONS IN TEMPORAL GAUGE

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ABSTRACT. This is a more or less straightforward adjustment of the paper arXiv:1512.05197 by the author for the 2+1 dimensional Maxwell-Klein-Gordon equations in temporal gauge to the 3+1 dimensional situation. They are shown to be locally well-posed for low regularity data even below energy level improving a result by Yuan. Fundamental for the proof is a partial null structure of the nonlinearity which allows to rely on bilinear estimates in wave-Sobolev spaces by d'Ancona, Foschi and Selberg, on an $(L_x^4 L_t^2)$ - estimate for the solution of the wave equation, and on the proof of a related result for the Yang-Mills equations by Tao.

1. INTRODUCTION AND MAIN RESULTS

Consider the Maxwell-Klein-Gordon equations

$$\partial^\alpha F_{\alpha\beta} = -Im(\phi \overline{D_\beta \phi}) \quad (1)$$

$$D^\mu D_\mu \phi = m^2 \phi \quad (2)$$

in Minkowski space $\mathbb{R}^{1+3} = \mathbb{R}_t \times \mathbb{R}_x^3$ with metric $diag(-1, 1, 1, 1)$. Greek indices run over $\{0, 1, 2, 3\}$, Latin indices over $\{1, 2, 3\}$, and the usual summation convention is used. Here $m \in \mathbb{R}$ and

$$\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}, A_\alpha : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}, F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, D_\mu = \partial_\mu + iA_\mu.$$

A_μ are the gauge potentials, $F_{\mu\nu}$ is the curvature. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (1) we obtain the Gauss-law constraint

$$\partial^j F_{j0} = -Im(\phi \overline{D_0 \phi}). \quad (3)$$

The system (1),(2) is invariant under the gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi, \phi \rightarrow \phi' = e^{i\chi} \phi, D_\mu \rightarrow D'_\mu = \partial_\mu + iA'_\mu.$$

This allows to impose an additional gauge condition. We exclusively consider the temporal gauge

$$A_0 = 0. \quad (4)$$

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In this gauge the system (1),(2) is equivalent to

$$\partial_t \partial^j A_j = \operatorname{Im}(\phi \overline{\partial_t \phi}) \quad (5)$$

$$\square A_j = \partial_j(\partial^k A_k) - \operatorname{Im}(\phi \overline{\partial_j \phi}) + A_j |\phi|^2 \quad (6)$$

$$\square \phi = -i(\partial^k A_k) \phi - 2iA^k \partial_k \phi + A^k A_k \phi + m^2 \phi, \quad (7)$$

where $\square = -\partial_t^2 + \Delta$ is the d'Alembert operator.

Other choices of the gauge are the Coulomb gauge $\partial^j A_j = 0$ and the Lorenz gauge $\partial^\mu A_\mu = 0$.

The classical (3+1)-dimensional Maxwell-Klein-Gordon system has been studied by Klainerman and Machedon [6] where the existence of global solutions for data in energy space and above in Coulomb gauge was shown. Uniqueness in a suitable subspace was also shown. For the temporal gauge they also showed a similar result by using a suitable gauge transformation applied to the solution constructed in Coulomb gauge. They made use of a null structure for the main bilinear term to achieve this result. Local well-posedness in Coulomb gauge for data for ϕ in the Sobolev space H^s and for A in H^r with $r = s > 1/2$, i.e., almost down to the critical space with respect to scaling, was shown by Machedon and Sterbenz [7]. Global well-posedness below energy space (for $r = s > \sqrt{3}/2$) in Coulomb gauge was shown by Keel, Roy and Tao [4].

The problem in Lorenz gauge was considered by Selberg and Tesfahun [13], who detected a null structure also in this case, and proved global well-posedness in energy space, especially also unconditional uniqueness in this space. The author [9] proved local well-posedness for $s = \frac{3}{4} + \epsilon$ and $r = \frac{1}{2} + \epsilon$.

The problem in temporal gauge was treated by Yuan [16] directly in $X^{s,b}$ -spaces. He stated local well-posedness in $X^{s,b}$ -spaces for large data for ϕ in H^s and for A in H^r with $r = s > 3/4$, where he just referred to the estimates given for Tao's small data local well-posedness results [15] in the Yang-Mills case. As a consequence he proved existence of a global solution in energy space and also uniqueness in subspaces of $X^{s,b}$ -type. Unconditional uniqueness in the natural solution space in the finite energy case was shown by the author [10]. These results in temporal gauge rely on a similar result by Tao [15] for the Yang-Mills equations and small data.

All these results were given in the (3+1)-dimensional case.

In 2+1 dimensions local well-posedness in Lorenz gauge for $s = \frac{3}{4} + \epsilon$ and $r = \frac{1}{4} + \epsilon$ was shown by the author [9]. In Coulomb gauge local well-posedness for $s = r = \frac{1}{2} + \epsilon$ and also for $s = \frac{5}{8} + \epsilon$, $r = \frac{1}{4} + \epsilon$ was obtained by Czubak and Pikula [3], which was slightly improved to the case $s = \frac{1}{2} + \epsilon$, $r = \frac{1}{4} + \epsilon$ in [11]. In the temporal gauge in [11] local well-posedness was shown for data under the minimal smoothness assumption $s = r = \frac{1}{2} + \frac{1}{12} + \epsilon$.

In the present paper we consider the (3+1)-dimensional case in the temporal gauge and lower down the minimal regularity assumptions on the data further using the same methods as in the (2+1)-dimensional case in [11]. In most of the cases one only has to adapt the parameters. We prove local well-posedness for data for ϕ in H^s and A in H^r , where $1 \geq s > \frac{3}{4}$ and $1 \geq r > \frac{1}{2}$, where uniqueness holds in $X^{s,b}$ spaces. The assumption $s, r \leq 1$ is just made to simplify the presentation by reducing to the most interesting cases and can certainly be significantly weakened so that especially smooth data are included. The critical case with respect to scaling is $r = s = \frac{1}{2}$, which we almost reach with respect to r . For technical reasons it is necessary to assume in a first step that the curl-free part of $A(0)$ vanishes (cf. Proposition 3.1). This condition is removed by a suitable gauge transformation

afterwards, which preserves the regularity of the solution. We need the null structure of some of the nonlinearities, the bilinear estimates for wave-Sobolev spaces $X_{|\tau|=|\xi|}^{s,b}$ by d'Ancona, Foschi and Selberg [1], and Tao's hybrid estimates [15] for the product of functions in wave-Sobolev spaces $X_{|\tau|=|\xi|}^{s,b}$ and in product Sobolev spaces $X_{\tau=0}^{l,b}$ (cf. the definition of the spaces below) which have to be generalized from the special case $l = s + \frac{1}{4}$. Moreover we need an appropriate generalization of the estimates for the terms which fulfill a null condition. Of fundamental importance is an $(L_x^4 L_t^2)$ - estimate for the solution of the wave equation which goes back to Tataru [5] and Tao [15].

We denote both the Fourier transform with respect to space and time and with respect to space by $\widehat{\cdot}$ or \mathcal{F} . The operator $|\nabla|^\alpha$ is defined by $(\mathcal{F}(|\nabla|^\alpha f))(\xi) = |\xi|^\alpha (\mathcal{F}f)(\xi)$ and similarly $\langle \nabla \rangle^\alpha$, where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. The inhomogeneous Sobolev spaces are denoted by $H^{s,p}$. For $p = 2$ we simply denote them by H^s . We repeatedly use the Sobolev embeddings $H^{s,p} \hookrightarrow L^q$ for $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{s}{3}$ and $1 < p \leq q < \infty$. We also use the notation $a \pm := a \pm \epsilon$ for a sufficiently small $\epsilon > 0$.

The standard space $X_\pm^{s,b}$ of Bourgain-Klainerman-Machedon type (which were already considered by M. Beals [2]) belonging to the half waves is the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^{3+1})$ with respect to the norm

$$\|u\|_{X_\pm^{s,b}} = \|\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \widehat{u}(\tau, \xi)\|_{L_{\tau\xi}^2}.$$

Similarly we define the wave-Sobolev space $X_{|\tau|=|\xi|}^{s,b}$ with norm

$$\|u\|_{X_{|\tau|=|\xi|}^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widehat{u}(\tau, \xi)\|_{L_{\tau\xi}^2}$$

and also $X_{\tau=0}^{s,b}$ with norm

$$\|u\|_{X_{\tau=0}^{s,b}} = \|\langle \xi \rangle^s \langle \tau \rangle^b \widehat{u}(\tau, \xi)\|_{L_{\tau\xi}^2}.$$

We also define $X_\pm^{s,b}[0, T]$ as the space of the restrictions of functions in $X_\pm^{s,b}$ to $[0, T] \times \mathbb{R}^3$ and similarly $X_{|\tau|=|\xi|}^{s,b}[0, T]$ and $X_{\tau=0}^{s,b}[0, T]$. We frequently use the estimate $\|u\|_{X_\pm^{s,b}} \leq \|u\|_{X_{|\tau|=|\xi|}^{s,b}}$ for $b \leq 0$ and the reverse estimate for $b \geq 0$.

We decompose $A = (A_1, A_2, A_3)$ into its divergence-free part A^{df} and its curl-free part A^{cf} :

$$A = A^{df} + A^{cf}, \quad (8)$$

where

$$A^{df} = (-\Delta)^{-1} \operatorname{curl} \operatorname{curl} A, \quad A^{cf} = -(-\Delta)^{-1} \nabla \operatorname{div} A. \quad (9)$$

Let $P = (-\Delta)^{-1} \operatorname{curl} \operatorname{curl}$ denote the projection operator onto the divergence free part. Then we obtain the equivalent system

$$\partial_t A^{cf} = -(-\Delta)^{-1} \nabla \operatorname{Im}(\phi \overline{\partial_t \phi}) \quad (10)$$

$$\square A^{df} = -P(\operatorname{Im}(\phi \overline{\nabla \phi}) + iA|\phi|^2) \quad (11)$$

$$\square \phi = i(\partial^j A_j^{cf})\phi + 2iA_j^{df} \partial^j \phi + 2iA_j^{cf} \partial^j \phi + A^j A_j \phi, \quad (12)$$

where A is replaced by (8).

Defining

$$\phi_\pm = \frac{1}{2}(\phi \pm i\langle \nabla \rangle^{-1} \partial_t \phi) \iff \phi = \phi_+ + \phi_-, \quad \partial_t \phi = i\langle \nabla \rangle(\phi_+ - \phi_-)$$

$$A_\pm^{df} = \frac{1}{2}(A^{df} \pm i\langle \nabla \rangle^{-1} \partial_t A^{df}) \iff A^{df} = A_+^{df} + A_-^{df}, \quad \partial_t A^{df} = i\langle \nabla \rangle(A_+^{df} - A_-^{df})$$

we can rewrite (10),(11),(12) as

$$\partial_t A^{cf} = -(-\Delta)^{-1} \nabla \operatorname{Im}(\phi \overline{\partial_t \phi}) \quad (13)$$

$$(-i\partial_t \pm \langle \nabla \rangle) A_{\pm}^{df} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S. \text{ of (11)} - A^{df}) \quad (14)$$

$$(-i\partial_t \pm \langle \nabla \rangle) \phi_{\pm} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S. \text{ of (12)} - \phi). \quad (15)$$

The initial data are transformed as follows:

$$\phi_{\pm}(0) = \frac{1}{2}(\phi(0) \pm i^{-1} \langle \nabla \rangle^{-1} (\partial_t \phi)(0)) \quad (16)$$

$$A_{\pm}^{df}(0) = \frac{1}{2}(A^{df}(0) \pm i^{-1} \langle \nabla \rangle^{-1} (\partial_t A^{df})(0)). \quad (17)$$

Our main result is preferably formulated in terms of the system (5),(6),(7).

Theorem 1.1. *1. Assume $1 \geq r > \frac{1}{2}$, $1 \geq s > \frac{3}{4}$, $s + \frac{1}{4} > l > 1$. Let $\phi_0 \in H^s(\mathbb{R}^3)$, $\phi_1 \in H^{s-1}(\mathbb{R}^3)$, $a_0 \in H^r(\mathbb{R}^3)$, $a_1 \in H^{r-1}(\mathbb{R}^3)$ be given, which satisfy the compatibility condition*

$$\partial_j a_1^j = \operatorname{Im}(\phi_0 \overline{\phi_1}) \quad (18)$$

Then there exists $T > 0$, such that (5),(6),(7) with initial conditions $\phi(0) = \phi_0$, $(\partial_t \phi)(0) = \phi_1$, $A(0) = a_0$, $(\partial_t A)(0) = a_1$ has a unique local solution

$$\phi = \phi_+ + \phi_- \quad , \quad A = A_+ + A_- + \tilde{A}$$

with

$$\phi_{\pm} \in X_{\pm}^{s, \frac{1}{2}+\epsilon}[0, T], \quad A_{\pm} \in X_{\pm}^{r, \frac{3}{2}-r+\epsilon}[0, T], \quad \tilde{A} \in X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}[0, T],$$

where $\epsilon > 0$ is sufficiently small.

2. This solution satisfies

$$\phi \in C^0([0, T], H^s(\mathbb{R}^3)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^3)),$$

$$A \in C^0([0, T], H^r(\mathbb{R}^3)) \cap C^1([0, T], H^{r-1}(\mathbb{R}^3)).$$

2. BASIC TOOLS

Fundamental for us are the following estimates. We frequently use the classical Sobolev multiplication law in dimension three :

$$\|uv\|_{H^{-s_0}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}, \quad (19)$$

if $s_0 + s_1 + s_2 \geq \frac{3}{2}$ and $s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2)$, where at most one of these inequalities is an equality.

The corresponding bilinear estimates in wave-Sobolev spaces were proven by d'Ancona, Foschi and Selberg in the three-dimensional case in [1] in a form which includes some more limit cases which we do not need.

Proposition 2.1. *For $s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R}$ and $u, v \in \mathcal{S}(\mathbb{R}^{3+1})$ the estimate*

$$\|uv\|_{X_{|\tau|=|\xi|}^{-s_0, -b_0}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{s_1, b_1}} \|v\|_{X_{|\tau|=|\xi|}^{s_2, b_2}}$$

holds, provided the following conditions are satisfied:

$$b_0 + b_1 + b_2 > \frac{1}{2}, \quad b_0 + b_1 \geq 0, \quad b_0 + b_2 \geq 0, \quad b_1 + b_2 \geq 0$$

$$\begin{aligned}
s_0 + s_1 + s_2 &> 2 - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 &> \frac{3}{2} - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2) \\
s_0 + s_1 + s_2 &> 1 - \min(b_0, b_1, b_2) \\
s_0 + s_1 + s_2 &> 1 \\
(s_0 + b_0) + 2s_1 + 2s_2 &> \frac{3}{2} \\
2s_0 + (s_1 + b_1) + 2s_2 &> \frac{3}{2} \\
2s_0 + 2s_1 + (s_2 + b_2) &> \frac{3}{2}
\end{aligned}$$

$$s_1 + s_2 \geq \max(0, -b_0), \quad s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).$$

Moreover we need the standard Strichartz estimate combined with the transfer principle (for a proof see [11], Theorem 8):

$$\|u\|_{L_{xt}^4} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}, \frac{1}{2}+}} \quad (20)$$

and the following estimate, which essentially goes back to Tataru [5] and Tao [14].

Lemma 2.1. *The following estimates hold:*

$$\begin{aligned}
\|u\|_{L_x^4 L_t^2} &\lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}, \frac{1}{2}+}}, \\
\|u\|_{L_x^4 L_t^{2+}} &\lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}+, \frac{1}{2}+}}.
\end{aligned}$$

Proof. One can simply refer to [14], Prop. 4.1. Alternatively by [5], Thm. B2 we obtain $\|\mathcal{F}_t u\|_{L_\tau^2 L_x^4} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{4}}}$, if $u = e^{it|\nabla|} u_0$ and \mathcal{F}_t denotes the Fourier transform with respect to time. This implies by Plancherel and Minkowski's inequality

$$\|u\|_{L_x^4 L_t^2} = \|\mathcal{F}_t u\|_{L_x^4 L_\tau^2} \leq \|\mathcal{F}_t u\|_{L_\tau^2 L_x^4} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{4}}}.$$

The transfer principle [11], Prop. 8 implies

$$\|u\|_{L_x^4 L_t^2} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}, \frac{1}{2}+}}. \quad (21)$$

Interpolation with (20) gives

$$\|u\|_{L_x^4 L_t^{2+}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}+, \frac{1}{2}+}}. \quad (22)$$

□

We now consider the Cauchy problem (13) - (17). Klainerman and Machedon detected that $A^{df} \cdot \nabla \phi$ and $P(\text{Im}(\phi \overline{\nabla \phi})_k)$ are null forms. An elementary calculation namely shows that

$$2A_i^{df} \partial^i \phi = Q_{ij}(\phi, |\nabla|^{-1}(R^i A^j - R^j A^i)) \quad (23)$$

and

$$P(\text{Im}(\phi \overline{\nabla \phi})_k) = -2R^j |\nabla|^{-1} Q_{kj}(Re \phi, Im \phi) \quad (24)$$

where the null form Q_{ij} is defined by

$$Q_{ij}(u, v) := \partial_i u \partial_j v - \partial_j u \partial_i v.$$

In order to estimate these null forms we also use the following estimate for the angle $\angle(\xi_1, \xi_2)$ between two vectors ξ_1 and ξ_2 .

Lemma 2.2. Assume $0 \leq \alpha, \beta, \gamma \leq \frac{1}{2}$ and $\xi_i \in \mathbb{R}^3$, $\tau_i \in \mathbb{R}$ ($i = 1, 2, 3$) with $\xi_1 + \xi_2 + \xi_3 = 0$, $\tau_1 + \tau_2 + \tau_3 = 0$. Then the following estimate holds for independent signs \pm and \pm' :

$$\angle(\pm\xi_1, \pm'\xi_2) \lesssim \left(\frac{\langle -\tau_1 \pm |\xi_1| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\alpha + \left(\frac{\langle -\tau_2 \pm' |\xi_2| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\beta + \left(\frac{\langle |\tau_3| - |\xi_3| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\gamma. \quad (25)$$

For a proof see for example [12], Lemma 2.1.

3. PROOF OF THEOREM 1.1

For the proof it is essential to show that we may assume in a first step that the initial data satisfy $a_0^{cf} = 0$ and that it is possible to cancel this condition in a second step by using a suitable gauge transformation.

Proposition 3.1. 1. Assume $1 \geq r > \frac{1}{2}$, $1 \geq s > \frac{3}{4}$, $s + \frac{1}{4} > l > 1$. Let $\phi_0 \in H^s(\mathbb{R}^3)$, $\phi_1 \in H^{s-1}(\mathbb{R}^3)$, $a_0 \in H^r(\mathbb{R}^3)$, $a_1 \in H^{r-1}(\mathbb{R}^3)$ be given, which satisfy the compatibility condition

$$\partial_j a_1^j = \text{Im}(\phi_0 \bar{\phi}_1)$$

and

$$a_0^{cf} = 0. \quad (26)$$

Then there exists $T > 0$, such that (10), (11), (12) with initial conditions $\phi(0) = \phi_0$, $(\partial_t \phi)(0) = \phi_1$, $A(0) = a_0$, $(\partial_t A)(0) = a_1$ has a unique local solution

$$\phi = \phi_+ + \phi_- \quad , \quad A = A_+^{df} + A_-^{df} + A^{cf}$$

with

$$\phi_\pm \in X_\pm^{s, \frac{1}{2}+\epsilon}[0, T], \quad A_\pm^{df} \in X_\pm^{r, \frac{3}{2}-r+\epsilon}[0, T], \quad A^{cf} \in X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}[0, T],$$

where $\epsilon > 0$ is sufficiently small.

2. This solution satisfies

$$\begin{aligned} \phi_\pm &\in C^0([0, T], H^s(\mathbb{R}^3)), \quad A_\pm^{df} \in C^0([0, T], H^r(\mathbb{R}^3)), \\ A^{cf} &\in C^0([0, T], H^l(\mathbb{R}^3)) \cap C^1([0, T], H^{l-1}(\mathbb{R}^3)). \end{aligned}$$

Proof of Proposition 3.1. Proof of part 2: We assume for the moment that part 1 is true. The compatibility condition (18), which is necessary in view of (3), determines a_1^{cf} as $a_1^{cf} = -(-\Delta)^{-1} \nabla(\text{Im}(\phi_0 \bar{\phi}_1))$.

It is not difficult to see that a_1^{cf} fulfills $a_1^{cf} \in H^{l-1}(\mathbb{R}^3)$. One only has to show that

$$\| |\nabla|^{-1}(\phi_0 \bar{\phi}_1) \|_{H^{l-1}} \lesssim \|\phi_0\|_{H^s} \|\phi_1\|_{H^{s-1}}.$$

By duality this is equivalent to

$$\|\phi_0 \phi_2\|_{H^{1-s}} \lesssim \|\phi_0\|_{H^s} \| |\nabla| \phi_2 \|_{H^{1-l}}.$$

In the case of high frequencies of ϕ_2 this follows from the Sobolev multiplication law (19) using $l \leq s + \frac{1}{4}$, and the low frequency case can be easily handled using $s > \frac{1}{2}$. In the same way we also obtain from (13): $\partial_t A^{cf} \in C^0([0, T], H^{l-1}(\mathbb{R}^3))$.

Proof of part 1: We use (23) and (24). By a contraction argument the local existence and uniqueness proof is reduced to suitable multilinear estimates for the right hand sides of (13), (14), (15). For (14), e.g., we make use of the following well-known estimate for a solution of the linear equation $(-i\partial_t \pm \langle \nabla \rangle) A_\pm^{df} = G$, namely

$$\|A_\pm^{df}\|_{X_\pm^{k,b}[0,T]} \lesssim \|A_\pm^{df}(0)\|_{H^k} + T^{b'-b} \|G\|_{X_\pm^{k,b'-1}[0,T]},$$

which holds for $k \in \mathbb{R}$, $\frac{1}{2} < b \leq b' < 1$ and $0 < T \leq 1$.

Thus the local existence and uniqueness for large data (in which case we have to choose $b < b'$), in the regularity class

$$\phi_{\pm} \in X_{\pm}^{s, \frac{1}{2}+\epsilon}[0, T], A_{\pm}^{df} \in X_{\pm}^{r, \frac{3}{2}-r+\epsilon}[0, T], A^{cf} \in X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}[0, T]$$

can be reduced to the following estimates for independent signs \pm, \pm', \pm'' , if we take the assumption $a^{cf} = 0$ into account (remark that we do not want to assume $a^{cf} \in H^l$ later):

$$\| |\nabla|^{-1}(\phi_1 \partial_t \phi_2) \|_{X_{\tau=0}^{l, -\frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (27)$$

$$\| |\nabla|^{-1} Q_{ij}(\phi_1, \phi_2) \|_{X_{\pm''}^{r-1, \frac{1}{2}-r+2\epsilon}} \lesssim \|\phi_1\|_{X_{\pm}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\pm'}^{s, \frac{1}{2}+\epsilon}}, \quad (28)$$

$$\| Q_{ij}(|\nabla|^{-1} \phi_1, \phi_2) \|_{X_{\pm''}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \|\phi_1\|_{X_{\pm}^{r, \frac{3}{2}-r+\epsilon}} \|\phi_2\|_{X_{\pm'}^{s, \frac{1}{2}+\epsilon}}, \quad (29)$$

$$\| \nabla A \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} + \| A \nabla \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \| A \|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (30)$$

$$\| A \phi_1 \phi_2 \|_{X_{|\tau|=|\xi|}^{r-1, \frac{1}{2}-r+2\epsilon}} \lesssim \min(\| A \|_{X_{|\tau|=|\xi|}^{r, \frac{3}{2}-r+\epsilon}}, \| A \|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}}) \prod_{i=1}^2 \|\phi_i\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (31)$$

$$\| A_1 A_2 \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \prod_{i=1}^2 \min(\| A_i \|_{X_{|\tau|=|\xi|}^{r, \frac{3}{2}-r+\epsilon}}, \| A_i \|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}}) \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \quad (32)$$

Proof of (29): The Fourier multiplier of $Q_{ij}(|\nabla|^{-1} \phi_1, \phi_2)$ is bounded by

$$\frac{|\xi_1 \times \xi_2|}{|\xi_1|} \lesssim |\xi_2| \angle(\pm \xi_1, \pm' \xi_2), \quad (33)$$

where $\xi_1 \times \xi_2$ denotes the cross product. If $\xi_1 + \xi_2 + \xi_3 = 0$ we also have

$$\frac{|\xi_1 \times \xi_2|}{|\xi_1|} = \frac{|\xi_1 \times \xi_3|}{|\xi_1|} \lesssim |\xi_3| \angle(\pm \xi_1, \pm'' \xi_3). \quad (34)$$

1. In the case $|\xi_3| \gtrsim \max(|\xi_1|, |\xi_2|)$ we use (33). It suffices to show

$$\begin{aligned} \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^s \langle -\tau_2 \pm' |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-s} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} \\ \cdot \angle(\pm \xi_1, \pm' \xi_2) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \end{aligned} \quad (35)$$

The Fourier transforms are nonnegative without loss of generality. Here $*$ denotes integration over $\sum_{i=1}^3 \xi_i = 0$, $\sum_{i=1}^3 \tau_i = 0$ and $d\xi d\tau = d\xi_1 d\xi_2 d\xi_3 d\tau_1 d\tau_2 d\tau_3$.

We use (25) with $\alpha = \beta = \frac{1}{2}$, $\gamma = \frac{1}{2}-$.

1.1. $|\xi_1| \leq |\xi_2|$. If the first term on the r.h.s. of (25) is dominant we use $|\xi_3| \sim |\xi_2|$ and reduce to

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{1-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

which follows from Prop. 2.1 for $r > \frac{1}{2}$, where we need the factor $\langle |\tau_1| - |\xi_1| \rangle^{1-r+}$ in the denominator. For the second and third term on the r.h.s. of (25) we only have to show

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \widehat{u}_2(\xi_2, \tau_2) \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

and

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \tau_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

respectively, both of which follow from Prop. 2.1 for $r > \frac{1}{2}$.

1.2. $|\xi_1| \geq |\xi_2|$. Using $|\xi_3| \sim |\xi_1|$ the l.h.s. of (35) is bounded by (up to a constant, what we tacitly allow hereafter)

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{1-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau$$

for the first term on the r.h.s. of (25). Similarly for the second and third term we obtain the bounds

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau$$

and

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau,$$

respectively, all of which are bounded by $\prod_{i=1}^3 \|u_i\|_{L_{xt}^2}$ for $r > \frac{1}{2}$ and $s > \frac{3}{4}$ by Prop. 2.1.

2. In the case $|\xi_3| \ll |\xi_1| \sim |\xi_2|$ we use (34). It suffices to show

$$\begin{aligned} \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3) |\xi_3|}{\langle \xi_3 \rangle^{1-s} \langle -\tau_3 \pm'' |\xi_3| \rangle^{\frac{1}{2}-}} \\ \cdot \angle(\pm \xi_1, \pm'' \xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (36) \end{aligned}$$

Using $|\xi_2| \gtrsim |\xi_3|$ and (25) with $\alpha = \beta = \frac{1}{2}$, $\gamma = \frac{1}{2}-$ and ξ_2 permuted with ξ_3 we bound the l.h.s. of (36) by

$$\begin{aligned} \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \widehat{u}_2(\xi_2, \tau_2) \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \\ + \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{2}-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \\ + \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{1-r+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau, \end{aligned}$$

which gives (36) by Prop. 2.1 and completes the proof of (29).

Proof of (28): We recall (33) and (34) and obtain the following bounds for the Fourier multiplier of $Q_{12}(\phi_1, \phi_2)$:

$$|\xi_1 \times \xi_2| \lesssim |\xi_1| |\xi_2| \angle(\pm \xi_1, \pm' \xi_2), \quad (37)$$

$$|\xi_1 \times \xi_2| \lesssim |\xi_1| |\xi_3| \angle(\pm \xi_1, \pm'' \xi_3). \quad (38)$$

1. In the case $|\xi_3| \gtrsim \max(|\xi_1|, |\xi_2|)$ we use (37) and reduce the desired estimate to

$$\begin{aligned} \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) |\xi_1|}{\langle \xi_1 \rangle^s \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^s \langle -\tau_2 \pm' |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{|\xi_3| \langle \xi_3 \rangle^{1-r} \langle |\tau_3| - |\xi_3| \rangle^{r-\frac{1}{2}-}} \\ \cdot \angle(\pm \xi_1, \pm' \xi_2) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (39) \end{aligned}$$

By symmetry we may assume $|\xi_1| \leq |\xi_2|$. We estimate $\angle(\pm\xi_1, \pm'\xi_2)$ by (25) with $\alpha = \beta = \frac{1}{2}$, $\gamma = r - \frac{1}{2} -$. We estimate the l.h.s. of (39) concerning the first term on the r.h.s. of (25) using $|\xi_3| \sim |\xi_2|$ by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-r+1} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{r-\frac{1}{2}-}} d\xi d\tau,$$

which gives (39) by Prop. 2.1, where we used $s > \frac{3}{4}$ and $r \leq 1$. For the second term on the r.h.s. of (25) we control the l.h.s. of (39) by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s+1-r}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{r-\frac{1}{2}-}} d\xi d\tau.$$

We apply Prop. 2.1 using $s > \frac{3}{4}$ and $1 \geq r > \frac{1}{2}$ to obtain (39). For the last term on the r.h.s. of (25) we estimate the l.h.s. of (39) using $|\xi_3| \sim |\xi_2| \gtrsim |\xi_1|$ as follows:

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-1+r-\frac{1}{2}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-r}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{2s-\frac{1}{2}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \end{aligned}$$

The last estimate follows from Prop. 2.1 using $s > \frac{3}{4}$ again.

2. In the case $|\xi_3| \ll |\xi_1| \sim |\xi_2|$ we use (38) and reduce the desired estimate to

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) |\xi_1|}{\langle \xi_1 \rangle^s \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-r} \langle -\tau_3 \pm'' |\xi_3| \rangle^{r-\frac{1}{2}-}} \\ & \cdot \angle(\pm\xi_1, \pm''\xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (40) \end{aligned}$$

We estimate $\angle(\pm\xi_1, \pm''\xi_3)$ by (25) with $\alpha = \beta = \frac{1}{2}$, $\gamma = r - \frac{1}{2} -$. We bound the l.h.s. of (40) for the first term on the r.h.s. of (25) (and similarly for the second term) using $|\xi_1| \sim |\xi_2|$ and $s \geq \frac{1}{2}$ by

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{3}{2}-r} \langle |\tau_3| - |\xi_3| \rangle^{r-\frac{1}{2}-}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{2s-r+\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{r-\frac{1}{2}-}} d\xi d\tau, \end{aligned}$$

which again implies (40) by Prop. 2.1.

For the last term on the r.h.s. of (25) we estimate the l.h.s. of (40) by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-}} d\xi d\tau.$$

We apply Prop. 2.1 using $s > \frac{3}{4}$, which implies (40) and completes the proof of (28).

Proof of (27): We first remark that the singularity of $|\nabla|^{-1}$ is harmless in three dimensions ([14], Cor. 8.2) and it can be replaced by $\langle \nabla \rangle^{-1}$. As a first step we use Sobolev's multiplication law (19) and obtain

$$\left| \int \int u_1 u_2 u_3 dx dt \right| \lesssim \|u_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|u_2\|_{X_{\tau=0}^{s, -\frac{1}{2}+\epsilon}} \|u_3\|_{X_{\tau=0}^{1-l, \frac{1}{2}-\epsilon}}$$

provided that $2s - \frac{1}{2} > l$ and $s > l - 1$, which is fulfilled under our assumptions. This implies taking the time derivative into account

$$\|\langle \nabla \rangle^{-1}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{l, -\frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}}. \quad (41)$$

In a second step we want to prove

$$\|\langle \nabla \rangle^{-1}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{l, -\frac{1}{2}+\epsilon}} + \|\langle \nabla \rangle^{-1}(\phi_2 \partial_t \phi_1)\|_{X_{\tau=0}^{l, -\frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}}. \quad (42)$$

If $\widehat{\phi_1}(\xi_3, \tau_3)$ is supported in $||\tau_3| - |\xi_3|| \gtrsim |\xi_3|$ we have the trivial bound

$$\|\phi_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (43)$$

so that (42) follows from (41). Assuming from now on $||\tau_3| - |\xi_3|| \ll |\xi_3|$ we have to prove

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2} \quad (44)$$

where

$$m = \frac{(|\tau_2| + |\tau_3|) \chi_{||\tau_3| - |\xi_3|| \ll |\xi_3|}}{\langle \xi_1 \rangle^{1-l} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^s \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}}.$$

Since $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$ and $\tau_1 + \tau_2 + \tau_3 = 0$ we have

$$|\tau_2| + |\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}-\epsilon}. \quad (45)$$

For the first term on the r.h.s. we have to show

$$\left| \int \int uvw dx dt \right| \lesssim \|u\|_{X_{\tau=0}^{1-l, 0}} \|v\|_{X_{\tau=0}^{s, 0}} \|w\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}},$$

which follows from Sobolev's multiplication law (19). For the other terms we use $l \geq 1$ so that $\langle \xi_1 \rangle^{l-1} \lesssim \langle \xi_2 \rangle^{l-1} + \langle \xi_3 \rangle^{l-1}$ and the second term on the r.h.s. reduces to the following estimates:

$$\left| \int \int uvw dx dt \right| \lesssim \|u\|_{X_{\tau=0}^{0, 0}} \|v\|_{X_{\tau=0}^{s+1-l, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}} \quad (46)$$

$$\left| \int \int uvw dx dt \right| \lesssim \|u\|_{X_{\tau=0}^{0, 0}} \|v\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-l+\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}}. \quad (47)$$

(46) follows from

$$\begin{aligned} \left| \int \int uvw dx dt \right| &\lesssim \|u\|_{L_x^2 L_t^2} \|v\|_{L_x^4 L_t^\infty} \|w\|_{L_x^4 L_t^2} \\ &\lesssim \|u\|_{X_{\tau=0}^{0, 0}} \|v\|_{X_{|\tau|=|\xi|}^{s+1-l, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}, \frac{1}{2}+\epsilon}}, \end{aligned}$$

where we used the Sobolev embedding $H_x^{s+1-l} \subset L_x^4$ under our assumption $s + \frac{1}{4} > l$ and Lemma 2.1. Similarly (47) follows using the embedding $H_x^s \subset L_x^4$ for $s > \frac{3}{4}$ and $s > l - \frac{1}{4}$ as well as Lemma 2.1. For the third term on the r.h.s. of (45) we have to show

$$\begin{aligned} \left| \int \int uvw dx dt \right| &\lesssim \|u\|_{X_{\tau=0}^{0, \frac{1}{2}-\epsilon}} \|v\|_{X_{\tau=0}^{s+1-l, 0}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon}} \\ \left| \int \int uvw dx dt \right| &\lesssim \|u\|_{X_{\tau=0}^{0, \frac{1}{2}-\epsilon}} \|v\|_{X_{\tau=0}^{s, 0}} \|w\|_{X_{|\tau|=|\xi|}^{s-l+\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon}}. \end{aligned}$$

Both estimates follow from

$$\left| \int \int uvw dx dt \right| \lesssim \|u\|_{L_x^2 L_t^\infty} \|v\|_{L_x^4 L_t^2} \|w\|_{L_x^4 L_t^{2+}}$$

using $H_x^{\frac{3}{4}} \subset L_x^4$, $s > \frac{3}{4}$, $s - l + \frac{1}{2} > \frac{1}{4}$ and Lemma 2.1.

We now come to the proof of (27) and remark that we may assume now that both functions ϕ_1 and ϕ_2 are supported in $||\tau| - |\xi|| \ll |\xi|$, because otherwise (27)

is an immediate consequence of (42) and (43). Thus (27) follows if we can prove the following estimate:

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

where

$$m = \frac{|\tau_3| \chi_{|\tau_2| - |\xi_2| \ll |\xi_2| \chi_{|\tau_3| - |\xi_3| \ll |\xi_3|}}}{\langle \xi_1 \rangle^{1-l} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}}.$$

Since $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$, $\langle \tau_2 \rangle \sim \langle \xi_2 \rangle$ and $\tau_1 + \tau_2 + \tau_3 = 0$ we obtain

$$|\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} + \langle \xi_2 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon}.$$

The first term is taken care of by the estimate

$$|\int \int uvw dx dt| \lesssim \|u\|_{X_{\tau=0}^{1-l,0}} \|v\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}},$$

which follows from Prop. 2.1 under our assumptions on s and l , as one easily checks.

In order to treat the second term on the right hand side we assume w.l.o.g. $|\xi_2| \geq |\xi_3|$, so that $\langle \xi_1 \rangle^{l-1} \lesssim \langle \xi_2 \rangle^{l-1}$, and estimate as follows:

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \langle \xi_1 \rangle^{l-1} \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{s-\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^{s-l+\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{s-\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau. \end{aligned}$$

Thus it remains to show

$$|\int \int uvw dx dt| \lesssim \|u\|_{X_{\tau=0}^{0, \frac{1}{2}-\epsilon}} \|v\|_{X_{|\tau|=|\xi|}^{s-l+\frac{1}{2}, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}}.$$

By Hölder's inequality we obtain

$$|\int \int uvw dx dt| \leq \|u\|_{L_x^2 L_t^{\infty-}} \|v\|_{L_x^4 L_t^2} \|w\|_{L_x^4 L_t^{2+}}.$$

For the last two factors we apply Lemma 2.1 using $s > \frac{3}{4}$ and $s > l - \frac{1}{4}$ and obtain the desired bound.

Proof of (30): This proof is similar to a related estimate for the Yang-Mills equation given by Tao [15]. We have to show

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

where

$$m = \frac{(|\xi_2| + |\xi_3|) \langle \xi_1 \rangle^{s-1}}{\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}-2\epsilon} \langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^l \langle \tau_3 \rangle^{\frac{1}{2}+\epsilon-}}.$$

Case 1: $|\xi_2| \lesssim |\xi_1|$ ($\Rightarrow |\xi_2| + |\xi_3| \lesssim |\xi_1|$).

We ignore the factor $\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}-2\epsilon}$ and use the averaging principle ([14], Prop. 5.1) to replace m by

$$m' = \frac{\langle \xi_1 \rangle^s \chi_{|\tau_2| - |\xi_2| \sim 1} \chi_{|\tau_3| \sim 1}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^l}.$$

Let now τ_2 be restricted to the region $\tau_2 = T + O(1)$ for some integer T . Then τ_1 is restricted to $\tau_1 = -T + O(1)$, because $\tau_1 + \tau_2 + \tau_3 = 0$, and ξ_2 is restricted to

$|\xi_2| = |T| + O(1)$. The τ_1 -regions are essentially disjoint for $T \in \mathbb{Z}$ and similarly the τ_2 -regions. Thus by Schur's test ([14], Lemma 3.11) we only have to show

$$\sup_{T \in \mathbb{Z}} \int_* \frac{\langle \xi_1 \rangle^s \chi_{\tau_1 = -T+O(1)} \chi_{\tau_2 = T+O(1)} \chi_{|\tau_3| \sim 1} \chi_{|\xi_2| = |T|+O(1)}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^l} \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau$$

$$\lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}.$$

The τ -behaviour of the integral is now trivial, thus we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\langle \xi_1 \rangle^s \chi_{|\xi_2| = T+O(1)}}{\langle T \rangle^s \langle \xi_3 \rangle^l} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}. \quad (48)$$

Assuming now $|\xi_3| \leq |\xi_1|$ (the other case being simpler) it only remains to consider the following two cases:

Case 1.1: $|\xi_1| \sim |\xi_3| \gtrsim T$. We now use our assumption $l \geq s$, so that it suffices to show

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\chi_{|\xi_2| = T+O(1)}}{T^l} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

The l.h.s. is bounded by

$$\begin{aligned} & \sup_{T \in \mathbb{N}} \frac{1}{T^l} \|f_1\|_{L^2} \|f_3\|_{L^2} \|\mathcal{F}^{-1}(\chi_{|\xi| = T+O(1)} \widehat{f}_2)\|_{L^\infty(\mathbb{R}^3)} \\ & \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^l} \|f_1\|_{L^2} \|f_3\|_{L^2} \|\chi_{|\xi| = T+O(1)} \widehat{f}_2\|_{L^1(\mathbb{R}^3)} \\ & \lesssim \sup_{T \in \mathbb{N}} \frac{T}{T^l} \prod_{i=1}^3 \|f_i\|_{L^2} \lesssim \prod_{i=1}^3 \|f_i\|_{L^2} \end{aligned}$$

for $l \geq 1$.

Case 1.2: $|\xi_1| \sim T \gtrsim |\xi_3|$. In this case it suffices to show

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\chi_{|\xi_2| = T+O(1)}}{\langle \xi_3 \rangle^l} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

An elementary calculation shows that the l.h.s. is bounded by

$$\sup_{T \in \mathbb{N}} \|\chi_{|\xi| = T+O(1)} * \langle \xi \rangle^{-2l} \|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} \prod_{i=1}^3 \|f_i\|_{L_x^2} \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2},$$

using that $l > 1$.

Case 2. $|\xi_1| \ll |\xi_2|$ ($\Rightarrow |\xi_2| + |\xi_3| \lesssim |\xi_2|$).

Exactly as in case 1 we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\langle T \rangle^{1-s} \chi_{|\xi_2| = T+O(1)}}{\langle \xi_1 \rangle^{1-s} \langle \xi_3 \rangle^l} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

Using $|\xi_3| \sim |\xi_2| \sim T \gg |\xi_1|$ and $l > 1$ we crudely estimate:

$$\frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} \langle \xi_3 \rangle^l} \sim \frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} T^l} \lesssim \frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} \langle \xi_1 \rangle^{s+T^{l-s-}}} \lesssim \frac{1}{\langle \xi_1 \rangle^{1+}}.$$

Thus we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\chi_{|\xi_2| = T+O(1)}}{\langle \xi_1 \rangle^{1+}} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2},$$

which can be shown as in Case 1.2. The proof of (30) is complete.

Proof of (31): We estimate by Sobolev's multiplication law (19) and Lemma 2.1:

$$\begin{aligned} \|A\phi_1\phi_2\|_{X_{|\tau|=|\xi|}^{r-1, \frac{1}{2}-r+2\epsilon}} &\lesssim \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} \lesssim \|A\|_{L_t^\infty H_x^r} \|\phi_1\phi_2\|_{L_t^2 H_x^{\frac{1}{2}+}} \\ &\lesssim \|A\|_{X_{|\tau|=|\xi|}^{r, \frac{1}{2}+\epsilon}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \end{aligned}$$

We also obtain for $r \leq 1$, $l \geq 1$ and $s > \frac{3}{4}$:

$$\begin{aligned} \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} &\lesssim \|A\phi_1\phi_2\|_{L_t^2 L_x^2} \lesssim \|A\|_{L_t^\infty L^6} \|\phi_1\|_{L_t^4 L_x^6} \|\phi_2\|_{L_t^4 L_x^6} \\ &\lesssim \|A\|_{X_{\tau=0}^{1, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{\frac{3}{4}, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{\frac{3}{4}, \frac{1}{2}+\epsilon}} \\ &\lesssim \|A\|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \end{aligned}$$

where we used that by Sobolev and Strichartz (20) we obtain

$$\|\phi_j\|_{L_t^4 L_x^6} \lesssim \|\phi_j\|_{L_t^4 H_x^{\frac{1}{4}, 4}} \lesssim \|\phi_j\|_{X_{|\tau|=|\xi|}^{\frac{3}{4}, \frac{1}{2}+\epsilon}}.$$

Proof of (32): By Sobolev's multiplication rule (19) and $l \geq 1$ we obtain

$$\begin{aligned} \|A_1 A_2 \phi\|_{L_t^2 H_x^{s-1}} &\lesssim \|A_1 A_2\|_{L_t^2 H_x^{\frac{1}{2}}} \|\phi\|_{L_t^\infty H_x^s} \lesssim \|A_1\|_{L_t^4 H_x^1} \|A_2\|_{L_t^4 H_x^1} \|\phi\|_{L_t^\infty H_x^s} \\ &\lesssim \|A_1\|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}} \|A_2\|_{X_{\tau=0}^{l, \frac{1}{2}+\epsilon-}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \end{aligned}$$

Next Lemma 2.1 implies:

$$\|A_1 A_2 \phi\|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+}} \lesssim \|A_1 A_2\|_{X_{|\tau|=|\xi|}^{0+, 0}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+}}.$$

Now using $r > \frac{1}{2}$ and Strichartz' estimate (20) we obtain

$$\begin{aligned} \|A_1 A_2\|_{X_{|\tau|=|\xi|}^{0+, 0}} &\lesssim \|A_1\|_{L_t^4 H_x^{0+, 4}} \|A_2\|_{L_t^4 H_x^{0+, 4}} \lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+}} \|A_2\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+}} \\ &\lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{r, \frac{1}{2}+}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{1}{2}+}}. \end{aligned}$$

Finally we also obtain

$$\begin{aligned} \|A_1 A_2\|_{X_{|\tau|=|\xi|}^{0+, 0}} &\lesssim \|A_1\|_{L_t^4 H_x^{0+, 4}} \|A_2\|_{L_t^4 H_x^{0+, 4}} \lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+}} \|A_2\|_{X_{\tau=0}^{1, \frac{1}{2}+}} \\ &\lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{r, \frac{1}{2}+}} \|A_2\|_{X_{\tau=0}^{l, \frac{1}{2}+}}. \end{aligned}$$

This completes the proof of (32) and part 1 of Proposition 3.1. \square

Now we eliminate the assumption $a_0^{cf} = 0$.

Proof of Theorem 1.1. We use Proposition 3.1 to construct a unique solution (ϕ', A') of the Cauchy problem for (10), (11), (12) with initial conditions $\phi'(0) = e^{-i\chi}\phi_0$, $(\partial_t \phi)(0) = e^{-i\chi}\phi_1$, $A'(0) = a_0^{df}$, $(\partial_t A)(0) = a_1$, where $a_0 \in H^r$, $a_1 \in H^{r-1}$, $\phi_0 \in H^s$, $\phi_1 \in H^{s-1}$ and the compatibility condition (18) is satisfied. Here $\chi := -(-\Delta)^{-1} \operatorname{div} a_0$ is chosen such that $\nabla \chi = a^{cf}(0)$. The assumptions for the data in Prop. 3.1 are now shown to be satisfied. It is immediately clear that $A'^{cf}(0) = 0$ and also $A'(0) \in H^r$, $(\partial_t A')(0) \in H^{r-1}$. In order to show the regularity of the data for ϕ' we start with the estimate

$$\|\nabla(uv)\|_{H^r} \leq c_1 \|\nabla u\|_{H^r} \|\nabla v\|_{H^r},$$

which holds for $r > \frac{1}{2}$ by (a variant of) (19). This implies

$$\begin{aligned} \|\nabla(e^{i\chi})\|_{H^r} &= \|\nabla(\sum_{k=0}^{\infty} \frac{(i\chi)^k}{k!})\|_{H^r} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\nabla(\chi^k)\|_{H^r} \\ &\leq \sum_{k=0}^{\infty} \frac{c_1^{k-1} \|\nabla\chi\|_{H^r}^k}{k!} = c_1^{-1} \exp(c_1 \|\nabla\chi\|_{H^r}) < \infty. \end{aligned} \quad (49)$$

Thus by (19) using $r > \frac{1}{2}$:

$$\|\phi'(0)\|_{H^s} = \|e^{i\chi}\phi_0\|_{H^s} \lesssim \|\nabla(e^{i\chi})\|_{H^r} \|\phi_0\|_{H^s} < \infty$$

and similarly also $(\partial_t \phi')(0) \in H^{s-1}$. The compatibility condition is also preserved, as one easily shows.

Consider now the gauge transformation

$$A'_\mu \rightarrow A_\mu = A'_\mu + \partial_\mu \chi, \quad \phi' \rightarrow \phi = e^{i\chi} \phi', \quad D'_\mu \rightarrow D_\mu = \partial_\mu + iA_\mu.$$

It certainly preserves the temporal gauge, because χ is independent of the time. This leads to a solution (A, ϕ) of (10),(11),(12) with initial conditions $A(0) = a_0^{df} + \nabla\chi = a_0^{df} + a_0^{cf} = a_0$, $(\partial_t A)(0) = a_1$, $\phi(0) = \phi_0$, $(\partial_t \phi)(0) = \phi_1$. What remains to be shown is that the regularity of the solution is preserved. It is easy to see that A has the same regularity as A' . Let now ψ be a smooth function with $\psi(t) = 1$ for $0 \leq t \leq T$ and $\psi(t) = 0$ for $t \geq 2T$. By Lemma 3.1 below and (49) we obtain for $s > \frac{3}{4}$ and $r > \frac{1}{2}$:

$$\begin{aligned} \|e^{i\chi} \phi'_\pm\|_{X_\pm^{s, \frac{1}{2}+\epsilon}_{[0,T]}} &\lesssim \|\nabla(e^{i\chi})\psi\|_{X_\pm^{r, \frac{1}{2}+\epsilon}} \|\phi'_\pm\|_{X_\pm^{s, \frac{1}{2}+\epsilon}_{[0,T]}} \\ &\lesssim \|\nabla(e^{i\chi})\|_{H^r} \|\phi'_\pm\|_{X_\pm^{s, \frac{1}{2}+\epsilon}_{[0,T]}} \\ &\lesssim c_1^{-1} \exp(c_1 \|a_0^{cf}\|_{H^r}) \|\phi'_\pm\|_{X_\pm^{s, \frac{1}{2}+\epsilon}_{[0,T]}} < \infty, \end{aligned}$$

so that the regularity of ϕ is also preserved. \square

In the last proof we used the following lemma.

Lemma 3.1. *The following estimate holds for $s > \frac{3}{4}$, $r > \frac{1}{2}$ and $\epsilon > 0$ sufficiently small:*

$$\|uv\|_{X_\pm^{s, \frac{1}{2}+\epsilon}} \lesssim \|\nabla u\|_{X_\pm^{r, \frac{1}{2}+\epsilon}} \|v\|_{X_\pm^{s, \frac{1}{2}+\epsilon}}.$$

Proof. By Tao [14], Cor. 8.2 we may replace ∇ by $\langle \nabla \rangle$ so that it suffices to prove

$$\|uv\|_{X_\pm^{s, \frac{1}{2}+\epsilon}} \lesssim \|u\|_{X_\pm^{r+1, \frac{1}{2}+\epsilon}} \|v\|_{X_\pm^{s, \frac{1}{2}+\epsilon}}.$$

We start with the elementary estimate

$$|(\tau_1 + \tau_2) \mp |\xi_1 + \xi_2|| \leq |\tau_1 \mp |\xi_1|| + |\tau_2 \mp |\xi_2|| + |\xi_1| + |\xi_2| - |\xi_1 + \xi_2|.$$

Assume now w.l.o.g. $|\xi_2| \geq |\xi_1|$. We have

$$|\xi_1| + |\xi_2| - |\xi_1 + \xi_2| \leq |\xi_1| + |\xi_2| + |\xi_1| - |\xi_2| = 2|\xi_1|,$$

so that

$$|(\tau_1 + \tau_2) \mp |\xi_1 + \xi_2|| \leq |\tau_1 \mp |\xi_1|| + |\tau_2 \mp |\xi_2|| + 2 \min(|\xi_1|, |\xi_2|).$$

Using Fourier transforms by standard arguments it thus suffices to show the following three estimates:

$$\begin{aligned}\|uv\|_{X_{\pm}^{s,0}} &\lesssim \|u\|_{X_{\pm}^{r+1,0}} \|v\|_{X_{\pm}^{s,\frac{1}{2}+\epsilon}} \\ \|uv\|_{X_{\pm}^{s,0}} &\lesssim \|u\|_{X_{\pm}^{r+1,\frac{1}{2}+\epsilon}} \|v\|_{X_{\pm}^{s,0}} \\ \|uv\|_{X_{\pm}^{s,0}} &\lesssim \|u\|_{X_{\pm}^{r+\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon}} \|v\|_{X_{\pm}^{s,\frac{1}{2}+\epsilon}}\end{aligned}$$

The first and second estimate easily follow from Sobolev, whereas the last one is implied by Lemma 2.1. \square

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